On Mathematical Analysis of Soil Structure using Consolidation Equations

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Abstract
This paper deals with an explicit finite difference solution for the one- and two-dimensional consolidation of a homogeneous clay layer. The finite difference method approximates the solution of a continuous problem by representing it in terms of a discrete set of elements such that there is an integer number of points in depth and an integer number of times at which we calculate the field variables; in this case, just the excess pore water pressure. The calculation of the average degree of consolidation is used as a medium for comparison between the numerical analysis and the empirical analysis. Here, we have solved two-dimensional consolidation equations numerically by using Alternating Direction Implicit (ADI) Method. Moreover, tridiagonal methods are used here alongside the ADI method. The main idea behind this technique is to avoid the complexities which usually occur while solving higher order partial differential equations. Finally, numerical examples are presented to show the relationship between the Pore Water Pressure (PWP) and Depth Time Grids (DTG). It was also discovered that the Average Degree of Consolidation (U_{ave}) directly varies with respect to the Time factor (T_v) as the time step increases.

Keywords: consolidation, finite difference, mathematical, soil structure, two-dimensional

Introduction

Consolidation is a process by which soils gain effective stress, through a dissipation of excess pore water pressure, and decrease in volume. However, sedimentation is the prior stage of the settlement of soils, where effective stress does not exist. These two phenomena are the fundamentals for the proper understanding of the sedimentation and consolidation processes in the containment. In fact, the void ratio, due to that effective stress, is controlled by the initial void ratio of the tailings (Bartholomeeusen, 2003; Been, 1980; Imai, 1981; Sills, 1998).

Many researchers have studied and explained the sedimentation process (Coe and Clevenger, 1916; Fitch, 1966; Kynch, 1952; Tan *et al*., 1988) and have also applied consolidation theory to soil sedimentation (Been and Sills, 1981; McRoberts and Nixon, 1976). Been (1980) found that slowed sedimentation could be derived from the consolidation theory by setting the effective stress to zero. Later, Schiffman (1982) stated that self-weight was a key component for consolidation while Mikasa and Takada (1984) demonstrated that the process commenced after sedimentation.

In general, large strain consolidation is associated with the process of sedimentation, when it is subjected to deposits below water (Koppula and Morgenstern, 1982). However, the sedimentation is rapid due to sub-aerial deposition and not taken into account explicitly in the model (Seneviratne *et al*., 1996). Therefore, the end of sedimentation and the starting of consolidation are usually chosen arbitrarily.

The theory was based on the assumptions of incompressible soil properties i.e., small strain, constant hydraulic conductivity and negligible self-weight (Terzaghi, 1943), which are not applicable for soft materials like tailings. The compressibility and hydraulic conductivity of tailings are highly non-linear. As a result, significant changes occur in settlement when it is subjected to a stress increment by continuous deposition and cannot be considered as a small strain problem. Later, it was found that incompressible soil properties were inappropriate (Davis and Raymond, 1965; Liu and Znidarčić, 1991) and that hydraulic conductivity had significant effects on changes to the void ratio. Additionally, self-weight is an important factor to distinguish between the sedimentation and consolidation phenomena of soft soils (Schiffman, 1982).

The two-dimensional consolidation theory with sand drains was proposed by Carillo (1942) and Barron (1948). A few decades later, Somogyi *et al*. (1984) derived a quasi two-dimensional finite strain consolidation model parallel to the one-dimensional derivation presented by Koppula (1970) providing an accurate estimation of the full-scale behaviour. Huerta and Rodriguez (1992) also presented a pseudo two-dimensional extension of the one-dimensional finite strain consolidation theory using the extended model to simulate the influence of the vertical drains. Bürger *et al*. (2004) described a two-dimensional analysis of sedimentation and consolidation in various shapes of a thickener, primarily used for dewatering of slurries, assuming the volumetric solids concentration was constant across each horizontal cross section. The simulation yielded a faster growth of sediment for the cone-shaped compared to the cylindrical-shaped containment. However, this approach differed from those of Somogyi *et al*. (1984) and Huerta and Rodriguez (1992) as the method did not consider the horizontal pore water flow. This process continues until the excess pore water pressure set up by an increase in total stress is completely dissipated (Craig, 2007). However, due to the low permeability of the soil, there will be a time lag between the application of the load and the extrusion of the pore water, and thus the settlement (Das, 2008). Consolidation is important in impervious soils, i.e., soils with low permeability, such as clay, whereas in sand, the dissipation of excess pore pressure is fast due to high permeability.

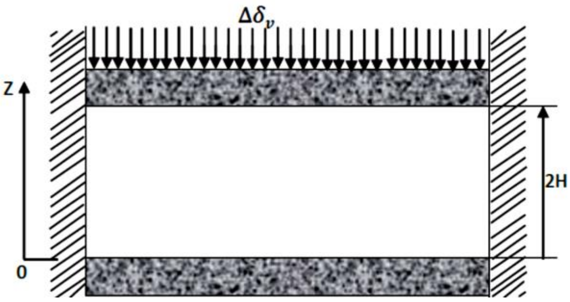

Figure 1: Example of a Clay layer drained on two faces (modified by Magnan, J. P., 1988)

Methods

Analytical Derivation and Solution of Two-Dimensional (2-D) Consolidation

The general equation of 2-D consolidation is:

$$\frac{\partial u}{dt} = C_x \frac{\partial^2 u}{dx^2} + C_z \frac{\partial^2 u}{dz^2} \quad (1)$$

where u = excess pore water pressure, C_x = coefficient of consolidation in horizontal direction, C_z = coefficient of consolidation in vertical direction, x = horizontal coordinate, z = vertical coordinate, and t = time (Craig, 2007).

Using integration by parts on both sides of equation (2) and employing variable separable approach yields

$$\Delta\bar{u} = exp\left[-\left(\frac{n\pi}{2H}\right)^2 (C_x + C_z)t\right] + \exp k$$
$$= kexp\left[-(C_x + C_z)\left(\frac{n\pi}{2H}\right)^2 t\right] \quad (3)$$

Applying Fourier series and recalling that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n cos\left(\frac{n\pi x}{l}\right) + b_n sin\left(\frac{n\pi x}{l}\right)\right] \quad (4)$$

Since the method used is half range of the sine Fourier transform based on the frequency (sinusoidal) of the interstitial pressure, the development into Fourier series will not affect the coefficient a_n.

Therefore, $a_0 = a_1 = a_2 = \dots a_{n-1} = a_n = 0$

Now, we have

$$F_s(x) = \sum_{n=1}^{\infty} b_n sin\left(\frac{n\pi x}{l}\right) \quad (5)$$

$$u(z,t) = \sum_{n=1}^{\infty} b_n sin\left(\frac{n\pi x}{l}\right) \quad (6)$$

where $b_n = \frac{1}{H}\int_0^{2H} f(z)\, sin\left(\frac{n\pi x}{2H}\right) dz$ (7)

Therefore, $\Delta u(z,t) =$

$$\sum_{n=1}^{\infty} kexp\left[-\left(\frac{n\pi}{2H}\right)^2 (C_x + C_z)t\right] sin\left(\frac{n\pi z}{2H}\right) \quad (8)$$

Boundary and Initial Conditions for Two-Dimensional (2-D) Consolidation Equation

The boundary conditions are not as fully prescribed in certain types of two-dimensional consolidation as in one-dimensional problems. It will be noticed that the boundary to the left and right in the compressible layer are not sharply defined in Figure 1. The termination of the calculation in these directions will be a consideration, which depends on the nature of the problem, the precision required and the judgment of the computer. However, it will generally be a simple matter to choose the number of significant figures desired and to terminate the calculations where values of less than half the last significant figure are encountered.

In certain cases, care must be taken, as the values will tend to spread outwards. This will occur where one of the boundary layers is impervious so that free drainage through its surface is prevented. Thus, the dissipation of hydrostatic excess pressure in the high-pressure regions can only be accomplished by the raising of pressure in the low-pressure regions by the flow of the pore water. This will result in swelling in those regions into which the water is flowing. If the coefficient of swelling is assumed to be equal to the

By applying sine Fourier transforms on equation (1), we have equation (2):

$$\int_0^{\infty} \frac{\partial\Delta u(z,t)}{\partial t} sin\left(\frac{n\pi z}{l}\right) dz = \int_0^{\infty} C_x \frac{\partial^2\Delta u(z,t)}{\partial x^2} sin\left(\frac{n\pi z}{l}\right) dz + \int_0^{\infty} C_z \frac{\partial^2\Delta u(z,t)}{\partial z^2} sin\left(\frac{n\pi z}{l}\right) dz \quad (2)$$

coefficient of consolidation, no extra labour is involved in the calculations but where a different value is assumed, or obtained from tests, the computation will be altered in regions where swelling is taking place as demonstrated in one of the previous works by R. F. Craig (2007).

Making the assumption that the soil is homogeneous and un-stratified, $C_x = C_z$.
Then equation (8) becomes

$$\Delta u(z,t) = \sum_{n=1}^{\infty} kexp\left[-2\left(\frac{n\pi}{2H}\right)^2 C_z t\right] sin\left(\frac{n\pi z}{2H}\right) \quad (9)$$

If we apply the conditions above, i.e., $\Delta u(0,t) = 0$, $\Delta u(2H,t) = 0$, and t = 0 into equation (8):

$$\Delta u(z,0) = \Delta\delta_v = \sum_{n=1}^{\infty} b_n sin\left(\frac{n\pi z}{2H}\right) \quad (10)$$

If we make identification of k = b_n, n = 1, 2, 3 …..
Then,

$$\Delta u(z,t) = \sum_{n=1}^{\infty} kexp\left[-2\left(\frac{n\pi}{2H}\right)^2 C_z t\right] sin\left(\frac{n\pi z}{2H}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{H}\int_0^{2H} f(z)\, sin\left(\frac{n\pi z}{2H}\right) dz\right) exp\left[-2\left(\frac{n\pi}{2H}\right)^2 C_z t\right] sin\left(\frac{n\pi z}{2H}\right)$$

Where $f(z) = \Delta\delta_v =$

$$\sum_{n=1}^{\infty}\left(\frac{1}{H}\int_0^{2H} \Delta\delta_v\, sin\left(\frac{n\pi z}{2H}\right) dz\right) exp\left[-2\left(\frac{n\pi}{2H}\right)^2 C_z t\right] sin\left(\frac{n\pi z}{2H}\right) \quad (11)$$

Then, we simplified b_n

$$b_n = \frac{1}{H}\int_0^{2H} \Delta\delta_v\, sin\left(\frac{n\pi z}{2H}\right) dz = \frac{\Delta\delta_v}{H}\left[\frac{-2H}{n\pi} cos\frac{n\pi z}{2H}\right]_0^{2H} = \frac{\Delta\delta_v}{H}\left(\frac{-2H}{n\pi}\right)(cosn\pi - 1) = \frac{-2\Delta\delta_v}{n\pi}(cosn\pi - 1) \quad (12)$$

When n is even, $b_n = 0$, and when n is odd, b_n will be valid

$$b_n = \frac{-2\Delta\delta_v}{n\pi}(-1-1);\ b_n = \frac{4\Delta\delta_v}{n\pi} \quad (13)$$

Putting equation (13) into (11) produces

$$\Delta u(z,t) = \sum_{n=1}^{\infty}\frac{4\Delta\delta_v}{n\pi} exp\left[-2\left(\frac{n\pi}{2H}\right)^2 C_z t\right] sin\left(\frac{n\pi z}{2H}\right) \quad (14)$$

Let n = 2m + 1 where m = 0

$$\bar{u} = \sum_{m=0}^{\infty}\frac{4\Delta\delta_v}{(2m+1)\pi} exp\left[-\frac{(2m+1)^2\pi^2}{2}\frac{C_z t}{H^2}\right] sin\left(\frac{n\pi z}{2H}\right) \quad (15)$$

Equation (15) is the analytical solution to the problem.

Results and Discussion

Numerical Solution of 2-D Consolidation using Alternating Direction Implicit (ADI) Method

In Mathematics, the Alternating Direction Implicit (ADI) Method is a finite difference method for solving parabolic and elliptic partial differential equations. It is most notably used to solve the problem of heat conduction or for solving the diffusion equations in two or more dimensions. The traditional method for solving the heat conduction equation is the Crank-Nicolson method. But the problem with Crank-Nicolson method is that the solution at each step of method is slower and a large memory scale is required to store the elements of the matrix. The advantage of ADI method is that the equations that have to be solved in every iteration have simpler structures and are thus easier to solve.

Applying ADI on equation (1), we obtain

$$\frac{u_{i,j}^{k+1} - u_{i,j}^{k}}{\Delta t} = c_z\left[\frac{u_{i+1,j}^{k} - 2u_{i,j}^{k} + u_{i-1,j}^{k}}{\Delta z^2}\right] + c_x\left[\frac{u_{i,j+1}^{k} - 2u_{i,j}^{k} + u_{i,j-1}^{k}}{\Delta x^2}\right]$$

$$u_{i,j}^{k+1} - u_{i,j}^{k} = \Delta t\left[c_z\left[\frac{u_{i+1,j}^{k} - 2u_{i,j}^{k} + u_{i-1,j}^{k}}{\Delta z^2}\right] + c_x\left[\frac{u_{i,j+1}^{k} - 2u_{i,j}^{k} + u_{i,j-1}^{k}}{\Delta x^2}\right]\right] \quad (16)$$

Assuming that the soil is homogeneous and unstratified; $c_z = c_x$, and for convenience, Δz can be taken to be equal to Δx.
Then

$$u_{i,j}^{k+1} - u_{i,j}^{k} = \frac{c_z\Delta t}{\Delta z^2}\left[u_{i+1,j}^{k} + u_{i-1,j}^{k} + u_{i,j+1}^{k} + u_{i,j-1}^{k} - 4u_{i,j}^{k}\right]$$

$$u_{i,j}^{k+1} = \frac{c_z\Delta t}{\Delta z^2}\left[u_{i+1,j}^{k} + u_{i-1,j}^{k} + u_{i,j+1}^{k} + u_{i,j-1}^{k} - 4u_{i,j}^{k}\right] + u_{i,j}^{k} \quad (17)$$

Let $\frac{c_z\Delta t}{\Delta z^2} = \beta$ then equation (17) becomes

$$u_{i,j}^{k+1} = \beta\left[u_{i+1,j}^{k} + u_{i-1,j}^{k} + u_{i,j+1}^{k} + u_{i,j-1}^{k} - 4u_{i,j}^{k}\right] + u_{i,j}^{k} \quad (18)$$

Equation (18) is the required numerical expression for 2-D consolidation theory.

In the ADI method the formula of equation (18) can be rearranged again by the following two ways:

$$u_{i,j}^{k+1} - u_{i,j}^{k} = \beta\left[(u_{i+1,j}^{k} + u_{i-1,j}^{k} - 2u_{i,j}^{k}) + (u_{i,j+1}^{k+1} + u_{i,j-1}^{k+1} - 2u_{i,j}^{k+1})\right] \quad (19)$$

$$u_{i,j}^{k+2} - u_{i,j}^{k+1} = \beta\left[(u_{i+1,j}^{k+2} + u_{i-1,j}^{k+2} - 2u_{i,j}^{k+2}) + (u_{i,j+1}^{k+1} + u_{i,j-1}^{k+1} - 2u_{i,j}^{k+1})\right] \quad (20)$$

Equation (19) is used to compute function values at all interval mesh points along columns while equation (20) is used to compute function values at all interval mesh points along rows. Note that for i = 1, 2, 3,..., n-1, equation (19) yields a tridiagonal system of equations and can be easily solved. Similarly, for j = 1, 2, 3, ..., n-1, equation (20) also yields a tridiagonal system of equations. In the ADI method equations (19) and (20) are used alternately. For example, for the first column, if i = 1, equation (19) gives:

$$u_{1,j}^{k+1} - u_{1,j}^{k} = \beta\left[(u_{2,j}^{k} + u_{0,j}^{k} - 2u_{1,j}^{k}) + (u_{1,j+1}^{k+1} + u_{1,j-1}^{k+1} - 2u_{1,j}^{k+1})\right];\ (j = 1,2,3,\ldots,n-1) \quad (21)$$

In order to apply the finite difference techniques in this research work, the problem treated by R. F. Craig (2007) is discussed and analysed as follows:

A half-closed clay layer (free-drainage at the upper boundary) is 10 m thick and the value of C_v is 7.9 m^2/year. The initial distribution of excess pore water pressure is as shown in Table 1.

Table 1: Initial Excess Pore-Water Pressure Distribution

Depth (m)	0	2	4	6	8	10
Pressure (kN/m^2)	60	54	41	29	19	15

*Source: Craig, R. F. (2007)

First time level when j = 0

$$u_{i,j+1} = \beta(u_{i+1,j} + u_{i-1,j}) + (1-2\beta)u_{i,j}$$

At $i = 1$, $u_{1,1} = \beta(u_{2,0} + u_{0,0}) + (1-2\beta)u_{1,0}$

$u_{1,1} = 0.1(u_{2,0} + u_{0,0}) + 0.8u_{1,0}$

$= 0.1(41 + 0) + 0.8 \times 54 = 47.30$

At $i = 2$, $u_{2,1} = 0.1(u_{3,0} + u_{1,0}) + 0.8u_{2,0} = 0.1(29 + 54) + 0.8 \times 41 = 41.10$

At $i = 3$, $u_{3,1} = 0.1(u_{4,0} + u_{2,0}) + 0.8u_{3,0} = 0.1(19 + 41) + 0.8 \times 29 = 29.20$

At $i = 4$, $u_{4,1} = 0.1(u_{5,0} + u_{3,0}) + 0.8u_{4,0} = 0.1(15 + 29) + 0.8 \times 19 = 19.60$

At $i = 5$, $u_{5,1} = 0.1(u_{6,0} + u_{4,0}) + 0.8u_{5,0} = 0.1(19 + 19) + 0.8 \times 15 = 15.80$

Second time level when j = 1

$$u_{i,j+1} = \beta(u_{i+1,j} + u_{i-1,j}) + (1-2\beta)u_{i,j}$$

At $i = 1$, $u_{1,2} = \beta(u_{2,1} + u_{0,1}) + 0.8u_{1,1}$

$u_{1,2} = 0.1(u_{2,1} + u_{0,1}) + 0.8u_{1,1}$

$= 0.1(41.1 + 0) + 0.8 \times 47.3$

$= 41.95$

At $i = 2$, $u_{2,2} = 0.1(u_{3,1} + u_{1,1}) + 0.8u_{2,1} = 0.1(29.2 + 47.3) + 0.8 \times 41.1 = 40.53$

At $i = 3$, $u_{3,2} = 0.1(u_{4,1} + u_{2,1}) + 0.8u_{3,1} = 0.1(19.6 + 41.1) + 0.8 \times 29.2 = 29.43$

At $i = 4$, $u_{4,2} = 0.1(u_{5,1} + u_{3,1}) + 0.8u_{4,1} = 0.1(15.8 + 29.2) + 0.8 \times 19.6 = 20.18$

At $i = 5$, $u_{5,2} = 0.1(u_{6,1} + u_{4,1}) + 0.8u_{5,1} = 0.1(19.6 + 19.6) + 0.8 \times 15.8 = 16.56$

Third time level when j = 2

$$u_{i,j+1} = \beta(u_{i+1,j} + u_{i-1,j}) + (1-2\beta)u_{i,j}$$

At $i = 1$, $u_{1,3} = 0.1(u_{2,2} + u_{0,2}) + 0.8u_{1,2} = 0.1(40.53 + 0) + 0.8 \times 41.95 = 37.61$

At $i = 2$, $u_{2,3} = 0.1(u_{3,2} + u_{1,2}) + 0.8u_{2,2} = 0.1(29.43 + 41.95) + 0.8 \times 40.53 = 39.56$

At $i = 3$, $u_{3,3} = 0.1(u_{4,2} + u_{2,2}) + 0.8u_{3,2} = 0.1(20.18 + 40.53) + 0.8 \times 29.43 = 29.62$

At $i = 4$, $u_{4,3} = 0.1(u_{5,2} + u_{3,2}) + 0.8u_{4,2} = 0.1(16.56 + 29.43) + 0.8 \times 20.18 = 20.74$

At $i = 5$, $u_{5,3} = 0.1(u_{6,2} + u_{4,2}) + 0.8u_{5,2} = 0.1(20.18 + 20.18) + 0.8 \times 16.56 = 17.28$

Fourth time level when j = 3

$$u_{i,j+1} = \beta(u_{i+1,j} + u_{i-1,j}) + (1-2\beta)u_{i,j}$$

At $i = 1$, $u_{1,4} = 0.1(u_{2,3} + u_{0,3}) + 0.8u_{1,3} = 0.1(39.56 + 0) + 0.8 \times 37.61 = 35.05$

At $i = 2$, $u_{2,4} = 0.1(u_{3,3} + u_{1,3}) + 0.8u_{2,3} = 0.1(29.62 + 37.61) + 0.8 \times 39.56 = 38.37$

At $i = 3$, $u_{3,4} = 0.1(u_{4,3} + u_{2,3}) + 0.8u_{3,3} = 0.1(20.74 + 39.56) + 0.8 \times 29.43 = 29.72$

At $i = 4$, $u_{4,4} = 0.1(u_{5,3} + u_{3,3}) + 0.8u_{4,3} = 0.1(17.28 + 29.62) + 0.8 \times 20.74 = 21.28$

At $i = 5$, $u_{5,4} = 0.1(u_{6,3} + u_{4,3}) + 0.8u_{5,3} = 0.1(20.74 + 20.74) + 0.8 \times 17.28 = 17.98$

Therefore, the results of the excess pore-water pressure distribution are presented in Table 2 for eleven of the twenty time steps.

Average Degree of Consolidation

To calculate the average degree of consolidation at the end of each time step, the enumerical integration of the equation is required (Craig, 2007):

$$U_{ave} = \left[1 - \left(\frac{\int_0^H \bar{u}_{t=1} dz}{\int_0^H \bar{u}_{t=0} dz}\right)\right] \times 100 \quad (22)$$

For the purpose of this research work, the Simpson one-third rule was adopted. Thus, the area under the curve of consideration is divided into **n**-number (where **n** is an even number), which ranges from *x* = *a* to *x* = *b*. Therefore, the area can be found using the

Table 2: Pore-Water Pressure and Depth Time Grids

	j	**0.00**	**0.05**	**0.01**	**0.20**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.80**	**0.90**	**1.00**
		0	0	0	0	0	0	0	0	0	0	0	0
i	**1**	54.00	47.30	41.95	34.05	28.57	24.58	21.57	19.23	17.35	15.82	14.56	13.50
	2	41.00	41.10	40.53	38.37	35.74	33.13	30.73	28.58	26.69	25.03	23.58	22.31
	3	29.00	29.20	29.43	29.72	29.68	29.35	28.82	28.17	27.47	26.75	26.03	25.34
	4	19.00	19.60	20.18	21.28	22.28	23.11	23.78	24.29	24.65	24.87	24.98	25.00
	5	15.00	15.80	16.56	17.98	19.27	20.44	21.47	22.36	23.09	23.68	24.12	24.43
Time (days)		0.00	18.25	36.50	73.00	109.50	146.0	182.50	219.00	255.50	292.00	328.50	365

expression:

$$\int_a^b \bar{u}dz = \frac{\Delta z}{3n}[u_0 + u_n + 4u_1 + 2u_2 + 4u_3 + 2u_4 + \cdots + 2u_{n-2} + 4u_{n-1}] \quad (23)$$

Where $\Delta z = \frac{b-a}{n}; H = b = 10, a = 0\ and\ n = 5$, $\Delta z = 2$, hence, we have

$$\int_0^H \bar{u}dz = \frac{1}{3}\Delta z\big[u_0 + u_n + 4(u_1 + u_3 + \cdots + u_{n-1}) + 2(u_2 + u_4 + \cdots + u_{n-2)}\big]$$

$$\int_0^{10} \bar{u}_{t=0}dz = \frac{1}{3}2[60.00 + 15.00 + 4(54.00 + 29.00) + 2(41.00 + 19.00)] = 351.333\text{KN/m}$$

At 1st term level

$$\int_0^{10} \bar{u}_{t=1}dz = \frac{1}{3}2[0 + 15.80 + 4(47.30 + 29.20) + 2(41.10 + 19.60)] = 295.467\text{kN/m}$$

At 2nd term level

$$\int_0^{10} \bar{u}_{t=2}dz = \frac{2}{3}[0 + 16.56 + 4(41.95 + 29.43) + 2(40.53 + 20.18)] = 282.333\text{kN/m}$$

⋮

At 20th term level

$$\int_0^{10} \bar{u}_{t=20}dz = \frac{2}{3}[0 + 24.43 + 4(13.50 + 25.34) + 2(22.31 + 25.00)] = 182.93\text{kN/m} \quad (24)$$

Table 3: Average Degree of Consolidation at all the twenty steps

Time step	Time *t* (years)	Time factor $T_v = \frac{C_v t}{H^2}$	Average Degree of Consolidation $U_{ave}(\%)$
1	0.05	0.00395	15.90
2	0.10	0.00790	19.64
3	0.15	0.01185	22.81
4	0.20	0.01580	25.55
5	0.25	0.01975	27.96
6	0.30	0.02370	30.11
7	0.35	0.02765	32.06
8	0.40	0.03160	33.84
9	0.45	0.03555	35.47
10	0.50	0.03950	36.99
11	0.55	0.04345	38.39
12	0.60	0.04740	39.71
13	0.65	0.05135	40.95
14	0.70	0.05530	42.11
15	0.75	0.05925	43.21
16	0.80	0.06320	44.25
17	0.85	0.06715	45.24
18	0.90	0.07110	46.18
19	0.95	0.07505	47.08
20	1.00	0.0790	47.93

By using equation (19), the typical equation for vertical column on the j^{th} will be solved as follows:

$$-\beta u_{i,j+1}^{k+1} + (1+2\beta)u_{i,j}^{k+1} - \beta u_{i,j-1}^{k+1} = \beta u_{i+1,j}^{k} + (1-2\beta)u_{i,j}^{k} + \beta u_{i-1,j}^{k}$$

Now by setting $i = 1, k = 0$, we have

$$-\beta u_{1,j+1}^{1} + (1+2\beta)u_{1,j}^{1} - \beta u_{1,j-1}^{1} = \beta u_{2,j}^{0} + (1-2\beta)u_{1,j}^{0} + \beta u_{0,j}^{0}$$

Setting $j = 1, 2, 3,$ then we have

$$u_{1,1}^1 = 52.9401, u_{1,2}^1 = 60.2817;\ u_{1,3}^1 = 65.4407 \quad (25)$$

For $i = 2, k = 0,$ we have

$$-\beta u_{2,j+1}^{1} + (1+2\beta)u_{2,j}^{1} - \beta u_{2,j-1}^{1} = \beta u_{3,j}^{0} + (1-2\beta)u_{2,j}^{0} + \beta u_{1,j}^{0}$$

Setting $j = 1, 2, 3,$ then we have

$$u_{2,1}^1 = 56.67; u_{2,2}^1 = 60.00; u_{2,3}^1 = 63.33 \quad (26)$$

For $i = 3, k = 0,$ we have

$$-\beta u_{3,j+1}^{1} + (1+2\beta)u_{3,j}^{1} - \beta u_{3,j-1}^{1} = \beta u_{4,j}^{0} + (1-2\beta)u_{3,j}^{0} + \beta u_{2,j}^{0}$$

Setting $j = 1, 2, 3,$ then we have

$$u_{3,1}^1 = 56.6667; u_{3,2}^1 = 60.0000;\ u_{3,3}^1 = 63.3333 \quad (27)$$

Computation of a Tridiagonal Matrix for Second Iteration

By using equation (20), the typical equation for horizontal row on i^{th} will be solved as follows:

$$u_{i,j}^{k+2} - u_{i,j}^{k+1} = \beta\big[(u_{i+1,j}^{k+2} + u_{i-1,j}^{k+2} - 2u_{i,j}^{k+2}) + (u_{i,j+1}^{k+1} + u_{i,j-1}^{k+1} - 2u_{i,j}^{k+1})\big] \quad (28)$$

Simplifying equation (28), we have

$$\beta u_{i+1,j}^{k+2} + (1+2\beta)u_{i,j}^{k+2} - \beta u_{i-1,j}^{k+2} = \beta u_{i,j+1}^{k+1} + (1-2\beta)u_{i,j}^{k+1} + \beta u_{i,j-1}^{k+1} \quad (29)$$

Now by setting $j = 1, k = 0$, we have

$$\beta u_{i+1,1}^{2} + (1+2\beta)u_{i,1}^{2} - \beta u_{i-1,1}^{2} = \beta u_{i,2}^{1} + (1-2\beta)u_{i,1}^{1} + \beta u_{i,0}^{1} \quad (30)$$

Setting $i = 1, 2, 3,$ then we have

$$-\beta u_{2,1}^{2} + (1+2\beta)u_{1,1}^{2} - \beta u_{0,1}^{2} = \beta u_{1,2}^{1} + (1-2\beta)u_{1,1}^{1} + \beta u_{1,0}^{1} \quad (31)$$

$$-\beta u_{3,1}^{2} + (1+2\beta)u_{2,1}^{2} - \beta u_{1,1}^{2} = \beta u_{2,2}^{1} + (1-2\beta)u_{2,1}^{1} + \beta u_{2,0}^{1} \quad (32)$$

$$-\beta u_{4,1}^{2} + (1+2\beta)u_{3,1}^{2} - \beta u_{2,1}^{2} = \beta u_{3,2}^{1} + (1-2\beta)u_{3,1}^{1} + \beta u_{3,0}^{1} \quad (33)$$

Substituting boundary conditions into equations (31), (32) and (33) with the value of $\beta = 0.1$:

$$-0.1u_{2,1}^{2} + 1.2u_{1,1}^{2} - 0.1u_{0,1}^{2} = 0.1u_{1,2}^{1} + 0.8u_{1,1}^{1} + 0.1u_{1,0}^{1} \quad (34)$$

$$-0.1u_{3,1}^{2} + 1.2u_{2,1}^{2} - 0.1u_{1,1}^{2} = 0.1u_{2,2}^{1} + 0.8u_{2,1}^{1} + 0.1u_{2,0}^{1} \quad (35)$$

$$-0.1u_{4,1}^{2} + 1.2u_{3,1}^{2} - 0.1u_{2,1}^{2} = 0.1u_{3,2}^{1} + 0.8u_{3,1}^{1} + 0.1u_{3,0}^{1} \quad (36)$$

In matrix form, equations (34), (35) and (36) become

$$\begin{bmatrix} 1.2 & -0.1 & 0 \\ -0.1 & 1.2 & -0.1 \\ 0 & -0.1 & 1.2 \end{bmatrix} \begin{bmatrix} u_{1,1}^2 \\ u_{2,1}^2 \\ u_{3,1}^2 \end{bmatrix} =$$

$$\begin{bmatrix} 0.1(60.2817) + 0.8(52.9401) + 0.1(10) + 0.1(25) \\ 0.1(60) + 0.8(56.6667) + 0.1(20) \\ 0.1(60) + 0.8(56.6667) + 0.1(30) + 0.1(50) \end{bmatrix}$$

Then,

$$\begin{bmatrix} 1.2 & -0.1 & 0 \\ -0.1 & 1.2 & -0.1 \\ 0 & -0.1 & 1.2 \end{bmatrix} \begin{bmatrix} u_{1,1}^2 \\ u_{2,1}^2 \\ u_{3,1}^2 \end{bmatrix} = \begin{bmatrix} 51.8803 \\ 53.3334 \\ 59.3334 \end{bmatrix} \qquad (37)$$

Applying elementary row-reduce operation in equation (37), we have

$$\begin{bmatrix} 1.2 & -0.1 & 0 \\ 0 & 14.3 & -1.2 \\ 0 & -0.1 & 1.2 \end{bmatrix} \begin{bmatrix} u_{1,1}^2 \\ u_{2,1}^2 \\ u_{3,1}^2 \end{bmatrix} = \begin{bmatrix} 51.8803 \\ 691.8811 \\ 59.3334 \end{bmatrix} R_2$$

$$= R_1 + 12R_2$$

Such that

$$\begin{bmatrix} 1.2 & -0.1 & 0 \\ 0 & 14.3 & -1.2 \\ 0 & 0 & 170.4 \end{bmatrix} \begin{bmatrix} u_{1,1}^2 \\ u_{2,1}^2 \\ u_{3,1}^2 \end{bmatrix} = \begin{bmatrix} 51.8803 \\ 691.8811 \\ 9176.5573 \end{bmatrix} R_3$$

$$= R_2 + 143R_3 \qquad (38)$$

Using backward substitution, equation (38) yields

$$u_{1,1}^2 = \frac{0.1(52.9024) + 51.8803}{1.2} = 47.6421$$

$$u_{2,1}^2 = \frac{1.2(53.8530) + 691.8811}{14.3} = 52.9024$$

$$u_{3,1}^2 = \frac{9176.5573}{170.4} = 53.8530$$

$u_{1,1}^2 = 47.6421$

$u_{2,1}^2 = 52.9024$

$u_{3,1}^2 = 53.8530$

For $j = 2, k = 0,$ we have

$$\beta u_{i+1,2}^2 + (1 + 2\beta)u_{i,2}^2 - \beta u_{i-1,2}^2 = \beta u_{i,3}^1 + (1 - 2\beta)u_{i,2}^1 + \beta u_{i,1}^1$$

Setting $i = 1, 2, 3,$ then we have

$u_{1,2}^2 = 60.4728$

$u_{2,2}^2 = 60.0397$

$u_{3,2}^2 = 60.0033$

For $j = 3, k = 0,$ we have

$$\beta u_{i+1,3}^2 + (1 + 2\beta)u_{i,3}^2 - \beta u_{i-1,3}^2 = \beta u_{i,4}^1 + (1 - 2\beta)u_{i,3}^1 + \beta u_{i,2}^1$$

Setting $i = 1, 2, 3,$ then we have

$u_{1,3}^2 = 69.6397$

$u_{2,3}^2 = 66.8695$

$u_{3,3}^2 = 66.1280$

Table 4a: First Iteration (time = 18.25 days)

0.000	10.0000	20.000	30.0000	40.0000
25.0000	52.9401	56.6667	56.6667	50.0000
65.0000	60.2817	60.0000	60.0000	60.0000
75.0000	65.4407	63.3333	63.3333	70.0000
120.0000	110.0000	100.0000	90.0000	80.0000

Table 4b: Second Iteration (time = 36.5 days)

0.000	10.0000	20.000	30.0000	40.0000
25.0000	47.6421	52.9024	53.8630	50.0000
65.0000	60.4728	60.0397	60.0033	60.0000
75.0000	69.6397	66.8695	66.1280	70.0000
120.0000	110.0000	100.0000	90.0000	80.0000

Table 4c: Third Iteration (time = 54.75 days)

0.000	10.0000	20.000	30.0000	40.0000
25.0000	44.1286	50.3371	51.9774	50.0000
65.0000	60.4499	59.3371	60.0029	60.0000
75.0000	72.4531	60.8379	67.9914	70.0000
120.0000	110.0000	100.0000	90.0000	80.0000

Table 4d: Fourth Iteration (time = 73.00 days)

0.000	10.0000	20.000	30.0000	40.0000
25.0000	41.3571	47.8073	50.3024	50.0000
65.0000	60.3349	58.8382	59.9025	60.0000
75.0000	74.2383	65.7846	69.1432	70.0000
120.0000	110.0000	100.0000	90.0000	80.0000

Table 4e: Fifth Iteration (time = 91.25 days)

0.000	10.0000	20.000	30.0000	40.0000
25.0000	39.4815	46.0803	49.0816	50.0000
65.0000	60.1166	58.8387	59.7525	60.0000
75.0000	75.4006	69.0414	69.8902	70.0000
120.0000	110.0000	100.0000	90.0000	80.0000

Table 4f: Sixth Iteration (time = 109.5 days)

0.000	10.0000	20.000	30.0000	40.0000
25.0000	37.9523	44.4588	48.0720	50.0000
65.0000	59.9669	58.7872	59.6483	60.0000
75.0000	76.6566	71.5580	70.8693	70.0000
120.0000	110.0000	100.0000	90.0000	80.0000

Table 4g: Seventh Iteration (time = 127.75 days)

0.000	10.0000	20.000	30.0000	40.0000
25.0000	36.6362	43.3811	47.3839	50.0000
65.0000	56.5574	58.8778	59.5721	60.0000
75.0000	78.5307	73.2390	71.5070	70.0000
120.0000	110.0000	100.0000	90.0000	80.0000

Table 4h: Eighth Iteration (time = 146 days)

0.000	10.0000	20.000	30.0000	40.0000
25.0000	35.5837	42.3569	46.7500	50.0000
65.0000	57.6131	58.7311	59.5166	60.0000
75.0000	78.4569	71.6687	72.1890	70.0000
120.0000	110.0000	100.0000	90.0000	80.0000

Table 4i: Ninth Iteration (time = 164.25 days)

0.000	10.0000	20.000	30.0000	40.0000
25.0000	35.0178	41.5832	46.3184	50.0000
65.0000	58.1863	57.8094	59.4641	60.0000
75.0000	78.5417	65.1504	72.3871	70.0000
120.0000	110.0000	100.0000	90.0000	80.0000

Table 4j: Tenth Iteration (time = 164.25 s)

0.000	10.0000	20.000	30.0000	40.0000
25.0000	34.5198	40.9087	45.9100	50.0000
65.0000	58.4392	57.2461	59.3053	60.0000
75.0000	78.3885	69.1425	72.3086	70.0000
120.0000	110.0000	100.0000	90.0000	80.0000

Conclusion

This study has presented the mathematical analysis of soil structures using two-dimensional consolidation equations. To model this phenomenon, the finite difference approach has been utilised to solve the problem.

The following were drawn out as the concluding part of this research work:

a. The procedure developed used the information in Table 1 for Pore Water Pressure (PWP) and Depth Time Grids (DTG) for the two-dimensional consolidation equation. Then, the finite difference technique, subjected to non-uniform initial excess pore water pressure distribution was employed, which gave excellent agreement with the work of R. L. Craig (2007). It was discovered that the degree of consolidation of any clay layer at a certain time depends upon the initial excess pore water pressure (Table 2).
b. The Average Degree of Consolidation (U_{ave}), using 20 steps was also investigated. The Average Degree of Consolidation (U_{ave}) directly varies with respect to the Time factor (T_v), as the time step (t) increases (Table 3).
c. The computation of the tridiagonal matrix, using 10 iterations (Tables 4a–4j), with time interval of 18.25 days showed that excellent results could be obtained by increasing the mesh refinement for both the time and the depth.
d. The degree of consolidation and excess pore water pressure depend widely upon the characteristics of the clay layer, such as coefficient of consolidation (C_v) and layer thickness (H).
e. The Alternating Direction Implicit (ADI) finite difference method is a very good method. It is convergent and unconditionally stable (Tables 4h–4j).
f. The results obtained in this work are in agreement with the existing ones especially the work of R. L. Craig (2007).

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